<u>LAST TIME:</u> For k a comm. ring. $\mathcal{D}(k)$ is a sym. mon. ∞-category.

Also for $A \in CAlg_k := \left\{ \begin{array}{l} \text{cdga} \\ \mathbb{E}_\infty\text{-ring.} \\ \text{simplicial comm ring.} \end{array} \right)$ (derived rings).

Since $CAlg_k^{disc.} := \tau_{\leq 0}(CAlg_k) \hookrightarrow CAlg_k$. k can be seen as a derived ring.

$\wr$ is

ordinary comm. k-algebra.

Moreover, the restriction of $(Spctr, Spctr)^{\otimes} : \Delta^{+,op} \to Cat_\infty$ to $Spctr^{\leq 0}$ in the $\Delta^{op} \hookrightarrow \Delta^{+,op}$ part of the source gives a module category: $(Spctr^{\leq 0}, Spctr)^{+} : \Delta^{+,op} \to Cat_\infty$, i.e. $Spctr$ is a $Spctr^{\leq 0}$-module. (Exercise: check this!)

<u>Def'n:</u> For any $A \in CAlg_k$ $(= Alg_{\mathbb{E}_\infty, k})$ let

$$Mod_A := Assoc\text{-}Mod((Spctr^{\leq 0}, Spctr)) \times_{AssocAlg(Spctr^{\leq 0})} \{A\}.$$

Technically, this would be left A-modules, but we don't need to worry about that since for us A will always be commutative hence left & right A-modules agree.

<u>Sanity check:</u> for $R \in CAlg_k^{disc}$ one has: $\mathcal{D}(R) \simeq Mod_R$.

To prove the sanity check claim we need a result that characterizes which stable ∞-cats. are obtained as $Mod A$ for some $A \in CAlg$.

<u>Thm:</u> [Schwede-Shipley]. Given $\mathcal{C}$ a stable ∞-category, then

$$\mathcal{C} \simeq Mod_A^{(right)} \quad \text{for} \quad A \in Assoc.Alg.(Spctr) \quad \text{iff:}$$

(i) $\mathcal{C}$ is presentable;
(ii) $\exists$ $C \in \mathcal{C}$ compact which generates $\mathcal{C}$, i.e.

$\forall\ D \in \mathcal{C}$ $\quad \mathrm{Ext}^n_{\mathcal{C}}(C, D) = 0 \quad \forall n \in \mathbb{Z} \Rightarrow D \simeq 0.$

$$\mathrm{Ext}^n_{\mathcal{C}}(C,D) = \mathrm{Hom}_{h\mathcal{C}}(C[-n], D) \quad \left(\simeq \pi_{-n} \mathrm{Hom}_{\mathcal{C}}(C, D) \text{ for } n<0 \right).$$

Moreover, in this case. $A \simeq \mathrm{End}_{\mathcal{C}}(C, C)$.

[Here $\mathrm{End}_{\mathcal{C}}(C,C)$ is a bit tricky to describe but it is a spectrum w/ the property that $\pi_i A = \mathrm{Ext}^{-i}_{\mathcal{C}}(C, C)$.]

Let's apply this theorem to our situation.

Consider $R \in \mathcal{D}(R)$, for any $M^* \in \mathcal{D}(R)$ one has:

$$H^i(M^*) = \mathrm{Ext}^i_{\mathcal{D}(R)}(R, M^*).$$

Since $H^i(M^*) = 0 \quad \forall i \in \mathbb{Z} \Rightarrow M^* = 0$ in $\mathcal{D}(R)$.

We obtain that R is a generator. It is also clear that R is compact, i.e. $\mathrm{Hom}_{\mathcal{D}(R)}(R, -)$ commutes w/ filtered colimits.

Let $A := \mathrm{End}_{\mathcal{D}(R)}(R, R)$, since $\mathrm{Ext}^{-i}_{\mathcal{D}(R)}(R,R) = 0$

unless $i=0$, and in that case we get $\pi_0(A) = R$.

One obtains. $\mathcal{D}(R) \simeq \mathrm{Mod}_R$.

Warning: The above argument only showed that the underlying ∞-cats. agree. One needs to be a bit more careful to check that the $\otimes$-structures agree. See [HA.7.1.2.7].

Consider $\underline{\mathrm{Ch}(k)}$ w/ its sym. mon. model structure:

$$\mathrm{CAlg}(\underline{\mathrm{Ch}}(k)) \underset{\text{oblv.}}{\overset{\mathrm{Sym}(-)}{\rightleftarrows}} \mathrm{Ch}(k).$$

There exists a model str. on $\mathrm{CAlg}(\underline{\mathrm{Ch}}(k))$ s.t. [can be promoted] is a Quillen adjunction.

Assume $k \supseteq \mathbb{Q}$!(*) The following thm. needs many restrictions on the model str. on the L.H.S. that's where the assu... is used.

Then by [HA, 4.5.4.7] one has an equivalence of ∞-categories: (*)

$$N(\mathrm{CAlg}(\underline{Ch}(k)_c))[W'^{-1}] \to \mathrm{CAlg}(N(\underline{Ch}(k)_c)[W^{-1}]).$$

$$\begin{matrix} \shortparallel & & \wr\wr \end{matrix}$$

∞-category underlying the sym. mon. model category whose objects are comm. differential graded algebras over k, i.e. com. alg. objects in the ordinary category $\underline{Ch}(k)$ of chain complexes.

$\mathrm{CAlg}(\mathcal{D}(k)) = \mathrm{CAlg}(\mathrm{Mod}_k)$.

Actually, we never discussed the sym. monoidal structure of Mod_A, here is a brief discussion.

First we need the relative tensor product.

Let ${}_A\mathrm{Mod}_B$ denote the category of A-B-bimodules.

[This can be defined by considering $\Delta^{+,+';op}$ where $[n]^{+,+'} = \{+' < 0 < 1 < \dots < n < +\}$ and associating A to $+'$ when acting on $+'[n]$ & B to $+$ when acting on $+[n]$ as one did before for left modules.

Equivalently, one has ${}_A\mathrm{Mod}_B \simeq {}_A\mathrm{Mod}(\mathrm{Mod}_B)$.]

Thm: [HA. 4.4.2.8] Given three associative algebras A, B & C one has a functor:

$$(-)\underset{B}{\otimes}(-) : {}_A\mathrm{Mod}_B \times {}_B\mathrm{Mod}_C \to {}_A\mathrm{Mod}_C \quad \text{which}$$

(i) is given by the Bar construction, i.e. $\forall N \in {}_A\mathrm{Mod}_B$, $\forall M \in {}_B\mathrm{Mod}_C$

$$N \underset{B}{\otimes} M \simeq \mathrm{colim}(\cdots \substack{\to\\\to\\\to} N \otimes B \otimes M \rightrightarrows N \otimes M).$$

(ii) $(-)\underset{B}{\otimes}(-)$ preserves geometric realizations on each factor, i.e. colimits indexed by Δ^{op}.

In particular, when A is commutative, one has an equivalence:

$$\mathrm{Mod}_A \simeq {}_A\mathrm{Mod}_A \quad \text{and this gives a functor:}$$

$(-) \underset{A}{\otimes} (-) : \text{Mod}_A \times \text{Mod}_A \to \text{Mod}_A$.

FACT: ① endows Mod_A w/ a the structure of a sym. mon. ∞-category. [HA, 4.5.2.1].

② For A discrete. $\text{Mod}_A \simeq \mathcal{D}(A)$ as $\otimes$-cats.

Moreover, given a map $f: A \to B$ in CAlg, one naturally has a functor.

$$\text{Mod}_A \leftarrow \text{Mod}_B : \text{oblv}_{B\to A}$$

given by forgetting the B-module structure to an A-mod. str.

FACT 2: $\text{oblv}_{B\to A}$ has a left adjoint given by $(-) \underset{A}{\otimes} B$, where B is seen as an A-mod via $\text{oblv}_{B\to A}$.

Properties of modules over a derived ring.

We will be useful to have the following subcategories of Mod_A.

$$\text{Perf}(A) \xrightarrow{[7.2.4.1]} \text{Mod}_A^{a.p.} \xrightarrow{(A \text{ connective})} \text{Mod}_A^- \subseteq \text{Mod}_A.$$

($\text{Mod}_A^{a.p.}$ stable.)

$\text{Perf}(A)$ = {smallest stable subcategory of Mod_A containing A & stable under retracts}.

$\text{Mod}_A^{a.p.}$ = $\{M \in \text{Mod}_A \mid M \in \text{Mod}_A^-$ & ~~∃k s.t. $M \in \text{Mod}_A$~~ $\forall n \geq 0$ $\tau^{\geq -n}(M)$ is compact in $\text{Mod}_A^{\geq -n}\}$

Mod_A^- = $\bigcup_{n\geq 0} \text{Mod}_A^{\leq n}$.

IS [HA, 7.2.4.2].

$\{M \in \text{Mod}_A \mid M$ is compact, i.e. $\text{Hom}(M,-)$ commutes w/ filt. colimits$\}$.

(⇓ [HTT, §5.3]. $\text{Mod}_A = \text{Ind}(\text{Perf}(A))$.)

$\text{Perf}(A)$

IS [HA, 7.2.4.4]

$\{M \in \text{Mod}_A \mid M$ is dualizable, i.e. $\exists M^\vee$ and maps u, c s.t.

$M \xrightarrow{\text{id}_M \otimes u} M \otimes M^\vee \otimes M \xrightarrow{c \otimes \text{id}_M} M$ &

$M^\vee \xrightarrow{\text{id}_{M^\vee} \otimes u} M^\vee \otimes M \otimes M^\vee \xrightarrow{c \otimes \text{id}_{M^\vee}} M^\vee$ are isom.$\}$.